U(1) charges in weakly-coupled free-fermionic heterotic string models

William Hicks*

Department of Physics, Brown University, Providence, RI 02912, USA

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Abstract

In order to explore the string landscape and in hopes of deriving phenomenologically realistic string models, a construction method for weakly-coupled free-fermionic heterotic string (WCFFHS) is described in detail, and an algorithm is presented for deriving properties of $U(1)$ groups in these models. Initial results from the construction of approximately 1.4 million gauge models are presented and analyzed, yielding information on the role of $U(1)$ charges in determining matter state uniqueness. The role of $U(1)$ groups (particularly anomalous $U(1)$ groups) in these models is considered in detail. Finally, some tools for dealing with anomalous $U(1)$ groups in models of this type are presented, and future avenues of research are discussed.

*William_Hicks@Brown.edu

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I. INTRODUCTION

Over the past several decades, theoreticians have developed a remarkably complete picture of known particle interactions into the framework known rather prosaically as the Standard Model. With experimentalists at the Large Hadron Collider narrowing in on the Higgs boson, the only remaining unobserved component of the Standard Model, it seems likely that all of the key predictions of the Standard Model will be experimentally verified by the middle of the decade [1]. Nevertheless, the Standard Model cannot give a complete picture of physics at this time, and it is equally likely that experimentalists will find signals indicating the presence of phenomena not explained by the Standard Model during the same time period.

Already, we can point to some examples of phenomena for which the Standard Model provides no explanation or prediction. Most notably, the Standard Model cannot describe gravitational interactions, and attempts to derive a complete quantum theory of gravity have so far failed. While this usually poses no particular difficulties due to the extremely low strength of gravitational interactions, such a theory would be necessary to describe systems which are both inherently quantum and extremely massive. Obvious examples of such systems include black holes and the universe during the earliest stages of the Big Bang.

Another experimentally verified departure from the Standard Model comes from neutrino-mixing. The observed mixing of lepton flavors for solar neutrinos implies that neutrinos must have mass, but the Standard Model makes no such prediction [2]. For theoreticians attempting to derive post-Standard Model physics, this provides a solid check against a verifiably non-Standard Model phenomenon.

Finally, while the Standard Model provides an accurate description of physical phenomena, it provides little explanation of why the properties of observed particles and interactions are what they are. For example, while the Standard Model describes the properties of three generations of quarks and requires at least three generations for consistency with experimental observations, it does not provide any fundamental justification for the existence of exactly three generations. More generally, the Standard Model contains over twenty arbitrary parameters which must be tuned to fit experimental observations rather than emerging naturally from an underlying theory. While it is entirely possible that these parameters have no more fundamental explanation, depending, perhaps, on the arbitrary initial conditions of the universe, this explanation is somewhat troublesome, and many theorists hope that a post-Standard Model theory will provide more basic justification of some or all of these parameters.

A. Basics of string theory

In light of these challenges, theoreticians have explored a number of post-Standard Model theories, but none have received so much attention or enjoyed so much success as string theory. Stated in its simplest form, the basic postulate of string theory is extremely straightforward: namely, string theory suggests that rather than being infinitely small points, the fundamental particles are actually extended one-dimensional objects known as strings, whose properties are determined by their oscillatory patterns. While this postulate seems on its surface to be relatively innocuous, it has profound implications. Most significantly, string theorists discovered early on that any consistent model of string theory includes a massless spin-2 particle: a graviton. Thus, while the Standard Model cannot consistently incorporate gravity, string theory immediately provides a gravitational description consistent with the rest of its interactions. In addition to its success with gravity, string theory offers some hope of providing more fundamental explanations for the properties described by the Standard Model, since the oscillatory modes of the strings determine those properties. By determining possible modes of oscillation, we can therefore derive those properties.

The process of determining these oscillatory modes proceeds in a fashion similar to that for any other harmonic oscillator. We begin by defining an action for the system, then apply the Euler-Lagrange equations to derive the equations of motion. There are several different types of string theory, and the details of this derivation varies among the different types. In order to examine some general results of this derivation, however, we begin by considering the bosonic string, the simplest and earliest-derived form of string theory.

To understand these results, we first require some general notation. Being a one-dimensional object, a string sweeps out a (1 + 1)-dimensional surface in spacetime, which we refer to as the worldsheet. The worldsheet may be parameterized by two coordinates (τ and σ), and the motion of the string may be completely described by the function $x^\mu(\tau, \sigma)$, where x^μ refers to standard spacetime coordinates. For convenience, we will hereafter replace τ and σ with the more general coordinates ξ^0 and ξ^1, which we can redefine to fit any convenient gauge. For the moment, we take $(\xi^0, \xi^1) = (\tau, \sigma)$. We can then express the action for the bosonic string as

$$S[x, \gamma] = -\frac{T}{2} \int d^2\xi \sqrt{-\gamma}\gamma^{ab}\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} \tag{1}$$

where T is the string tension, γ^{ab} is a metric on the worldsheet (with γ being its determinate), and $\eta_{\mu\nu}$ is the Minkowski metric [3]. This formulation is known as the Polyakov action. Analyzing this action, however, we obtain a somewhat disturbing result. The general solutions for the equations of motion derived from this action allows for particle states with negative norm, which can in turn lead to a theory with nonsensical probabilities (to be further discussed in section V A). In order to eliminate such states, we must apply the so-called "Virasoro constraints," which require that the components of the worldsheet energy-momentum tensor go to zero. Examining these constraints, however, we see that they only yield physical particle state solutions in exactly 26 spacetime dimensions. In other words, bosonic string theory *requires* the existence of 22 additional spatial dimensions to yield a physically sensible spectrum of particle states. These additional dimensions serve to cancel the conformal anomaly that would otherwise invalidate the theory.

For many physicists, this surprising result was enough to eliminate string theory as a viable model, but the existence of these additional dimensions is not necessarily in conflict with our everyday observations of 4 spacetime dimensions. As first explored in Kaluza-Klein theory, it is possible to introduce additional spatial dimensions if they are rolled up or "compactified" into a manifold small enough that we cannot explore it at everyday energy scales. Thus, any attempt at building a successful string theory model must include some compactification scheme for these extra dimensions.

B. Types of string theory

As it turns out, the simple bosonic string is insufficient to describe the particles that we see. Most obviously, the bosonic string does not admit spacetime fermions as solutions for physical states. In order to produce such states, we turn instead to the Ramond-Neveu-Schwarz (RNS) formulation of the supersymmetric string or "superstring." In this formulation, we introduce an additional Majorana spinor (ψ) on the worldsheet for every spacetime dimension, which yields the following action [3]:

$$S = -\frac{T}{2} \int d^2\xi \left(\partial_a x^\mu \partial^a x_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu \right). \tag{2}$$

Performing the same analysis as for the bosonic string, we find that canceling the conformal anomaly for the RNS superstring requires 10 spacetime dimensions.

Combining this with our analysis of bosonic strings, we can now describe the five types of string theory [4]:

- **Type I:** Strings in this type of theory are superstrings and may be described using the RNS formulation. They differ from Type II strings primarily in that they are unoriented (i.e. their states do not change if the orientation of the superstring is reversed [5]). Furthermore, these superstrings can be either open or closed (forming a loop).

- **Type IIA and Type IIB:**These are oriented superstrings and may also be described via the RNS formulation. They differ in a choice of how the space of physical states is truncated leading either to a chiral (IIB) or non-chiral (IIA) theory. These strings are all closed.

- **Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$** These theories consist of closed strings that are hybrids of Type I and bosonic strings. Oscillations moving in one direction (left-moving or counterclockwise) are treated as oscillations of the bosonic string, while oscillations moving in the other direction (right-moving or clockwise) are treated as oscillations of the superstring. The two types differ in their overall gauge group.

While the existence of five different valid string theories at first appeared somewhat disheartening, work by a number of theorists in the mid-90's demonstrated that they were in fact merely five different realms of a single underlying theory, connected by two types of duality [3]. T-duality demonstrates that under the transformation

$$R \to \frac{1}{R}, \tag{3}$$

where R is the compactification radius, Type IIA string models become Type IIB and vice versa. Furthermore we have the same equivalence between the two types of heterotic models. S-duality, which establishes equivalences based on the transformation

$$g_s \to \frac{1}{g_s} \tag{4}$$

where g_s is the string coupling constant, also relates Type I models to heterotic $SO(32)$ models. Similar dualities with respect to R and g_s relate M-theory, an 11-dimensional theory involving higher-dimensional membranes to Type IIA and heterotic $E_8 \times E_8$, so we can see that all of these theories are simply realizations of a larger theory under certain values of R and g_s. Nevertheless, distinguishing among these theories allows us to develop tools which are most useful in their various domains of applicability.

At first sight, the heterotic string is by far the most peculiar of these types of theories, and it is also the type which we consider most extensively in this paper. The most immediate peculiarity of such strings is the fact that they seem to simultaneously require 10 and 26 spacetime dimensions, depending on the direction of oscillation. This apparent paradox is resolved by allowing the 16 additional dimensions of the left-movers to form an even self-dual lattice. The two possible such lattices correspond to the two possible gauge groups.

C. The string landscape

After recognizing the equivalence of the five string theories, the task of experimentally testing string theory seems at first to be distinctly manageable: simply construct the spectrum of possible oscillator states and compare the properties of these states to observed particles. Unfortunately, this is not as straightforward as we might hope. Specifically, although we can construct the particle spectrum from the vacuum determined by the action, the criteria for selecting possible compactifications does not lead to a single consistent vacuum. In fact, research in the early 2000's established that the number of stable and consistent vacua, though finite, could be on the order of 10^{500} [6–8]. Although some theorists hold out hope for further selection criteria to reduce this so called "string landscape" to a somewhat more manageable number, there is no reason to believe at this time that any more precise criteria exist.

This problem is one of the greatest obstacles to unleashing the full predictive power of string theory, and a number of approaches have been proposed to circumvent it. One particular approach, adopted primarily by a collaboration known as the String Vacuum Project, is to construct and catalog a wide range of possible string models. By gathering statistical information on such models, we can perhaps gain insight into different features of the entire landscape, or even more excitingly, derive one or more string vacua consistent with the Standard Model. This requires, however, fast construction algorithms in order to build enough models to tell us something about the features of the landscape in its entirety. In light of this challenge, as well as this opportunity, we consider one particular type of string model, the weakly-coupled free-fermionic heterotic string (WCFFHS), and examine methods of constructing such models.

II. CONSTRUCTING WCFFHS STRING MODELS

A. Weakly-coupled free-fermionic heterotic strings

As described in section I B, heterotic strings are a hybrid between Type I and bosonic strings. As such, their left-movers move in 26 spacetime dimensions, while the right-movers move in 10 dimensions. In order to formally describe these dimensions in WCFFHS models, the degrees of freedom on each side will be fermionized. In 10 large spacetime dimensions, this gives us 32 real fermions on the left (two for each of the 16 extra bosonic degrees of freedom) and 8 real fermions for the 8 directions transverse to the string[3]. For each compactified dimension, we introduce one additional bosonic degree of freedom on each side. Fermionizing the bosonic degrees of freedom (yielding two fermions for each boson), we see that we have

$$\text{Right: } 2(14 - D) \tag{5}$$

$$\text{Left: } 2(26 - D) \tag{6}$$

fermionic degrees of freedom in a model with D large spacetime dimensions.

In general, these worldsheet fermions could interact via so-called Thirring interactions, leading to complications when constructing a given model. In particular, if we examine the action for a worldsheet boson X^μ, we will see terms of the form [9]

$$\partial_a X^\mu \partial^a X_\mu. \tag{7}$$

Expressed as fermions (ψ), this gives us

$$i\psi^{*\mu}\partial_a\psi_\mu + i\bar{\psi}^{*\mu}\partial_a\bar{\psi}_\mu - h\psi^*\psi\bar{\psi}^*\bar{\psi}, \tag{8}$$

where h is the Thirring coupling, which depends on the radius of compactifiction R for the model as follows [9]:

$$h = \frac{R}{2} - \frac{1}{2R}. \tag{9}$$

Thus, we see that at the self-dual radius $\left(R = \frac{1}{R}\right)$, h vanishes and we have no Thirring interactions. Such models are called free-fermionic models for obvious reasons, and we will confine our analysis to models of this type. We will also restrict our attention to models with weak string coupling ($g_s << 1$).

While the general construction of string spectra is a complex problem, in 1986 two groups independently developed a completely general and systematic construction method for fermionic heterotic strings compactified on tori [10, 11]. In [10], Kawai, Lewellen, and Tye describe two distinct but entirely equivalent systems of constructing heterotic string models, one of which is based on the physical oscillator states of the string and the other of which describes those states in terms of their fermionic charges. While the first method gives a more intuitive understanding of the origin of these states, the second is far more convenient for most tasks in constructing the spectrum and identifiying its properties. We will therefore begin by describing the oscillator construction as laid out in [10] and then proceed to the charge lattice construction for details of the construction process.

B. The oscillator construction[12]

The oscillator construction begins by describing the geometry of the compactification manifold via the boundary conditions on individual world-sheet fermions. A particular boundary condition can be specified by a number v, which specifies the phase a given fermion ψ^l picks up when transported around a non-contractible loop in spacetime:

$$\psi^l \to e^{-2\pi i v}\psi^l. \tag{10}$$

In general, real fermions in these models can be paired to form complex fermions, but as we shall see, such pairings are not always possible. For real fermions, v can only take on values 0 and $\frac{1}{2}$, but complex fermions may take on any rational value in the range $[0, 1)$.

In order to completely specify the geometry of the worldsheet, we choose a linearly independent set of basis vectors $\{\mathbf{W}_i\}$ These basis vectors span a sublattice of the space of possible $\mathbf{W}$-vectors which uniquely specifies the worldsheet geometry for a specific model. In particular, they will be used to generate a set of vectors $\{\overline{\alpha\mathbf{W}}\}$ whose components give the boundary conditions for each of the worldsheet fermions. Explicitly, this set of vectors, which specifies the sectors in our model, is generated by

$$\overline{\alpha\mathbf{W}} = \overline{\sum_i \alpha_i \mathbf{W}_i} \qquad \alpha_i \in [0, m_i) \cap \mathbb{Z} \tag{11}$$

where m_i is the smallest integer such that all elements of $m_i\mathbf{W}_i$ are integers and the overline notation indicates that we have taken each of the elements mod 1. In 10 large spacetime dimensions, we will have $(8 + 32)$-dimensional vectors to describe the boundary conditions on the 8 right-moving and 32 left-moving real fermions. In a complex basis, we could describe the geometry with half as many components on each side.

1. *Constraints on the partition function*

While these vectors $\{\mathbf{W}_i\}$ can describe the boundary conditions for an arbitrary model, we are not entirely free in our choice of such vectors. Some worldsheet configurations are not permitted, since we must ensure that a given geometry provides a consistent partition function for the entire model. In particular, we must ensure the following properties for the partition function:

- **Reparameterization invariance:** The partition function should be invariant under any valid reparameterization of the worldsheet coordinates ξ^0 and ξ^1
- **Worldsheet supersymmetry:** This, in combination with reparameterization invariance ensures positivity, and in the light-cone gauge, also gives us spacetime Lorentz invariance.

- **Superconformal invariance and absence of local 2D gravitational anomalies:** These requirements are (quite naturally) necessary for any physically sensible theory.

- **Modular invariance:** The partition function should be invariant under modular transformations of the worldsheet. In general, such a transformation will change the boundary conditions for any given fermion, but since modular transformations give valid parameterizations of a single torus, they should not change the overall contribution of a world-sheet fermion to the partition function.

- **Valid projection of physical states:** In general, constructing all possible oscillator states on a string vacuum will produce extra states that, if included in the model, would prevent the construction of a valid root structure for the model and lead to contributions to the partition function that are inconsistent with proper spin-statistics. Because of this, we must have a projection of states onto some physical space that ensures fermionic states contribute negatively to the partition function while bosons contribute positively.

Of these conditions, we will be most concerned with the last two, since modular invariance creates a strict set of rules for sensible boundary basis vectors, and the final condition requires that we also provide a valid Gliozzi-Scherk-Olive (GSO) projection matrix (with elements k_{ij} for our model) [13]. This matrix will be used to remove some of the states from the model and thereby ensure a valid gauge structure using the mechanism to be indicated in Equation 18. Worldsheet supersymmetry is ensured by the fermionization procedure for the degrees of freedom in the model, and while the other conditions determine the exact form of the overall partition function, they do not put any manifest constraints on the form of our basis vectors or GSO projection.

2. Modular invariance

Acknowledging these conditions, our task is now to construct a set of basis vectors as well as a GSO coefficient matrix consistent with these requirements. Unfortunately, these conditions are not as restrictive as we might hope, and they allow for the creation of an enormous number of possible string spectra. Nevertheless, in order to derive the explicit restrictions on the basis vectors, we consider the contribution of the boundary conditions of each fermion to the overall partition function under modular transformations.

Modular transformations of the worldsheet will be of the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \{a, b, c, d\} \in \mathbb{Z} \qquad ad - bc = 1 \tag{12}$$

and can be generated by the following elements:

$$\tau \to -\frac{1}{\tau} \tag{13}$$

$$\tau \to \tau + 1. \tag{14}$$

We can therefore vastly simplify our task of finding restrictions on $\{\mathbf{W}_i\}$ and k_{ij} by simply examining how the partition function transforms under these generators and demanding that $\{\mathbf{W}_i\}$ and k_{ij} collectively be chosen to ensure that the partition function remains invariant. These conditions are explicitly derived in [10], but here we will simply quote the results:

$$k_{ij} + k_{ji} = \mathbf{W}_i \cdot \mathbf{W}_j \pmod 1 \tag{15}$$

$$m_j k_{ij} = 0 \pmod 1 \tag{16}$$

$$k_{ii} + k_{i0} + s_i - \frac{1}{2}\mathbf{W}_i \cdot \mathbf{W}_i = 0 \pmod 1 \tag{17}$$

where the dot product carries a negative sign for the contribution from right-moving elements as well as a factor of $\frac{1}{2}$ for real fermions, and s_i is a parameter giving information about spacetime statistics for the model. s_i takes on value $\frac{1}{2}$ if both the first and second right-moving components (which correspond to non-compact directions and which will hereafter be referred to as the spacetime fermionic components) of a given basis vector have antiperiodic boundary conditions and value 0 if these components both have periodic boundary conditions.

3. Deriving the low-energy spectrum

Once we have found a valid set of basis vectors and GSO coefficients, we have everything we need to specify a particular string model, and we can then derive the physical spectrum. We will express possible oscillator states as vectors $\mathbf{N}_{\overline{\alpha\mathbf{W}}}$, where each element gives the fermionic occupation numbers for each of the world-sheet fermions of the physical state. Note that these states are labeled by their sectors $(\overline{\alpha\mathbf{W}})$, since they indicate excitations on the distinct vacuum states associated with each sector. In addition, we can specify $D-2$ non-compact bosonic excitations (referred to hereafter as spacetime bosonic excitations) on each side for the physical states where D is the number of large spacetime dimensions. The spectrum-generating equation is given by the GSO-projection condition:

$$\mathbf{W}_i \cdot \mathbf{N}_{\overline{\alpha \mathbf{W}}} = \sum_j k_{ij}\alpha_j + s_i + k_{0i} - \mathbf{W}_i \cdot \overline{\alpha \mathbf{W}} \tag{18}$$

where the dot product carries the same convention as in Equation 15 and Equation 17. With this expression, we can now generate all states which are in the state space projected onto via our chosen GSO-projection. By choosing modular invariant $\{\mathbf{W}_i\}$ and k_{ij} and using a proper fermionization procedure, we have also ensured that most of the constraints on our partition function have been satisfied. However, if we wish to make contact with the Standard Model, we must consider what other conditions are necessary for a given physical state to be included in the low-energy effective field theory (LEEFT).

In particular, we must consider what we know about the mass of the generated physical states. Since these states are built at the string energy scale, any massive states will be too massive to observe at everyday energies. Furthermore, we shall see that the mass for the right-moving and left-moving parts of the string are determined separately, and these masses must be consistent for a state to be included in the model at all. This is known as the level-matching condition.

4. Mass of the physical states

In order to examine the mass of the states in our model, we begin by analyzing the energy of the vacuum state associated with each sector. For a given boundary condition v on a complex worldsheet fermion, its energy contribution to the associated vacuum-state is given by

$$E_v = \frac{1}{2}\left(v^2 - v + \frac{1}{6}\right). \tag{19}$$

Real fermion boundary conditions will contribute half this value. The overall energy of the vacuum state for the left (right)-movers will be given by

$$E^{\text{left(right)}}_{\overline{\alpha \mathbf{W}}} = \sum_{l:\text{left(right)}} \left\{E_{\overline{\alpha \mathbf{W}}^l}\right\} - \frac{D-2}{24} \tag{20}$$

After determining the energy of the vacuum state, we then examine the contribution of each excitation of the physical state to its overall mass. In order to do so, it is convenient to break the fermionic occupation number down in order to account for its normal mode excitations separately. Splitting apart the fermionic occupation number, we can examine the fermion numbers for its normal mode excitations (n and $\overline{n}$) separately, where

$$\mathbf{N}^{l}_{\overline{\alpha\mathbf{W}}} = \sum_{q} n^{l}_{q} - \overline{n}^{l}_{q} \tag{21}$$

and q ranges over integer quanta of excitation. The mass m squared for the left(right)-moving part of a given state is then given by the following:

$$\begin{aligned} m^2\left(\text{left(right)}, n, \overline{n}, \tilde{M}\right) = & \sum_{l:\text{left(right)}} \Big\{ E_{\overline{\alpha\mathbf{W}}^{l}} \\ & + \sum_{q=1}^{\infty} \left[\left(q - \overline{\alpha\mathbf{W}}^{l}\right) \overline{n}^{l}_{q} + \left(q + \overline{\alpha\mathbf{W}}^{l} - 1\right) n^{l}_{q} \right] \Big\} \\ & - \frac{D-2}{24} + \sum_{l=1}^{D-2} \sum_{q=1}^{\infty} q \tilde{M}^{i}_{q} \end{aligned} \tag{22}$$

where $\tilde{M}$ refers to the spacetime bosonic excitations. Using Equation 22, we can now check the level-matching condition to ensure that a state is physical, as well as the masslessness condition to determine whether or not a given physical state will be present in the LEEFT. This is the last tool necessary to determine the particle spectrum for any model of this type.

In addition to determining the mass of a given state, we can determine its fermionic charge vector via

$$\mathbf{Q} = \overline{\alpha W} - \mathbf{W}_0 + \mathbf{N}_{\overline{\alpha W}}, \tag{23}$$

a formula which will provide an important correspondence between the oscillator and charge lattice constructions.

5. *A simple example*

We now consider the simplest possible example of a model for 10 large dimensions in this construction and derive its low-energy physical spectrum. Consider the basis vector

$$\mathbf{W}_0 = \left(\left(\frac{1}{2}\right)^{8} || \left(\frac{1}{2}\right)^{32} \right) \tag{24}$$

where the || notation separates the right-moving and left-moving part respectively. It turns out that modular invariance considerations guarantee that this particular basis vector appear in every $\{\mathbf{W}_i\}$ if we are to obtain a model with a non-zero partition function, so let us consider the model generated by $\mathbf{W}_0$ as its only basis vector.

Our first task is to determine valid values for our k_{ij} matrix, which in this case consists of the single element k_{00}. Applying the constraints in Equations 15-17, we obtain

$$2k_{00} = 3 \pmod 1 \tag{25}$$

$$2k_{00} = 0 \pmod 1 \tag{26}$$

$$2k_{00} + \frac{1}{2} - \frac{3}{2} = 0 \pmod 1. \tag{27}$$

This reduces to a single constraint ($2k_{00} = 0 \pmod 1$), which gives us $k_{00} = 0, \frac{1}{2}$. While both of these choices for our k_{ij} matrix yield the same model for this case, it is possible to obtain different models simply by varying the choice of GSO coefficients, and it is generally necessary to vary both $\{\mathbf{W}_i\}$ and k_{ij} to explore all possible models. For convenience, we will take $k_{00} = 0$.

Next, we determine the sectors (which in this case is rather trivially $\mathbf{W}_0$ itself and $\mathbf{0}$) and calculate the vacuum state energy for each sector by applying Equation 20 to the left- and right-moving parts separately. Recalling that we are currently working with real fermions (though we could just as easily deal with complex fermions by pairing the given fermions on each side), this gives us a left-moving vacuum state energy of -1 and a right-moving vacuum state energy of $-\frac{1}{2}$ for the $\mathbf{W}_0$-sector, which we denote by $\left[-\frac{1}{2}, -1\right]$. For the $\mathbf{0}$-sector, we have vacuum state energy $[0, 1]$. Since adding excitations on the vacuum state can only increase the mass, we see that the $\mathbf{0}$-sector cannot produce any massless states.

Having obtained the vacuum state energies, we consider the lowest-mass physical states that can be produced in this model. Applying Equation 18 to the $\mathbf{W}_0$-sector, we obtain the following constraint for the physical states:

$$\mathbf{W}_0 \cdot \mathbf{N}_{\mathbf{W}_0} = \frac{1}{2} \pmod 1. \tag{28}$$

Recognizing that we can ignore sign-contributions since $-\frac{1}{2} = \frac{1}{2} \pmod 1$, this in turn gives us (in a real basis)[14]

$$\frac{1}{2} \sum_{l=1}^{40} \sum_{q=1}^{\infty} n_l^q = \frac{1}{2} \pmod 1. \tag{29}$$

From this and the level-matching condition, we see that our lowest-mass physical state would have a single left-moving fermionic excitation, yielding 32 tachyons with $m^2 = -\frac{1}{2}$. Our next-lowest set of states are massless and have one fermionic excitation on the right. On the left, we could have either two fermionic excitations or one spacetime bosonic excitations, giving us gauge bosons and members of the gravity supermultiplet respectively. While we will address the issue of identifying gauge groups more explicitly in our discussion of the charge lattice construction, it is easy to see from the forms of the state vectors that the gauge bosons in this model form a representation of $SO(32)$. Any further states in this model will clearly be massive, so we now have our entire low-energy spectrum. While the presence of tachyons in this model is somewhat troubling, we can easily project out those states through the addition of a single new basis vector

$$\mathbf{W}_1 = \left((0)^8 \,||\, \left(\frac{1}{2}\right)^{32} \right) \tag{30}$$

which will remove any states with a single left-moving fermionic excitation via Equation 18 and generate 496 massless fermions from the $\mathbf{W}_1 + \mathbf{W}_0$ sector.

C. Charge lattice construction

Although the oscillator construction provides a fairly direct picture of the origin of the physical states in a string model, it proves inconvenient for deriving certain properties of the constructed model as well as numerical construction of a large number of string models. For such tasks, we will instead turn to the charge lattice construction, an entirely equivalent method in which we construct the fermionic charge vectors of the states directly rather than constructing their fermion occupation number vectors. While the results from section II B were almost completely general for fermionic heterotic string models, we will strictly confine our attention in this section to four-dimensional weakly-coupled free-fermionic heterotic models. Since our ultimate goal is to develop tools that can be used for exploration of the string landscape through rapid construction of models (and also to avoid any possible confusion with the oscillator construction), we will shift from the notation employed in section II B to that of [15], which presents a fast, robust, and highly extensible computational framework for generating WCFFHS models. We can then use this construction method to more readily explore some of the details of these models that would be more difficult to extract using the oscillator construction.

1. The basis vectors

As before, our construction method begins by specifying a set of basis vectors, which specify information about the geometry of the worldsheet. In this case, however, we will use substantially different conventions for how this information is specified, and we will refer to these basis vectors as the set $\{\vec{\alpha}_i^B\}$ to emphasize this distinction. We will begin by offering some additional notation for the components of the basic vectors before offering some constraints on allowed basis vector sets. We will then proceed to clarify their physical meaning after using them to construct the sectors in our model, noting here only that they serve the same function as the set $\{\mathbf{W}_i\}$ and that their components will determine the phases of worldsheet fermions.

It is convenient to provide some additional notation for the worldsheet fermions described in our model, as this will provide us with a set of labels for the various components of each $\vec{\alpha}_i^B$[16]. The right-moving part of a given basis vector $\vec{\alpha}^B$ will be denoted by the following components:

$$\big((\psi^1, \psi_c^1), (x, y, w)^1, (x, y, w)^2, (x, y, w)^3, (x, y, w)^4, (x, y, w)^5, (x, y, w)^6\big) \tag{31}$$

where ψ^1 and ψ_c^1 refer to non-compact fermionic modes (again referred to as spacetime fermionic modes) whose charges will determine whether a given state is a spacetime fermion, boson, or scalar. The left-moving part is split into three parts, the observable, hidden, and compactified sectors, which are labeled as

$$\text{Observable:}\quad \Big(\overline{\psi}^1, \overline{\psi}_c^1, \overline{\psi}^2, \overline{\psi}_c^2, \overline{\psi}^3, \overline{\psi}_c^3, \overline{\psi}^4, \overline{\psi}_c^4, \overline{\psi}^5, \overline{\psi}_c^5, \overline{\eta}^1, \overline{\eta}_c^1, \overline{\eta}^2, \overline{\eta}_c^2, \overline{\eta}^3, \overline{\eta}_c^3\Big) \tag{32}$$

$$\text{Hidden:}\quad \Big(\overline{\phi}^1, \overline{\phi}_c^1, \overline{\phi}^2, \overline{\phi}_c^2, \overline{\phi}^3, \overline{\phi}_c^3, \overline{\phi}^4, \overline{\phi}_c^4, \overline{\phi}^5, \overline{\phi}_c^5, \overline{\phi}^6, \overline{\phi}_c^6, \overline{\phi}^7, \overline{\phi}_c^7, \overline{\phi}^8, \overline{\phi}_c^8\Big) \tag{33}$$

$$\text{Compactified:}\quad (\overline{y}^1, \overline{y}^2, \overline{y}^3, \overline{y}^4, \overline{y}^5, \overline{y}^6, \overline{w}^1, \overline{w}^2, \overline{w}^3, \overline{w}^4, \overline{w}^5, \overline{w}^6) \tag{34}$$

Using this notation, we now turn to the problem of choosing valid basis vectors.

Each $\vec{\alpha}_i^B$ has rational components in the range $(-1, 1]$, and we associate with each such basis vector an integer N_i known as the order of that basis vector, which is given by the least common denominator of the elements of $\vec{\alpha}_i^B$. This will conveniently allow us to encode the numerator and denominator of elements in the basis vectors separately as integers for the purposes of programming. As with the vectors $\{\mathbf{W}_i\}$, we are not entirely free in how the choice of basis vectors, and we present the following rules to ensure a consistent, worldsheet-supersymmetric model:

- The vector $\mathbb{1}$ consisting of all 1's appears in every model
- The set $\{\vec{\alpha}_i^B\}$ should be linearly independent. This ensures that the $\vec{0}$-sector cannot be produced via a linear combination of basis vectors (with nonzero linear coefficients).
- The number of periodic components in each triplet (x, y, w) among the right-moving components must be odd. This ensures that the fermion that this triplet represents has proper spin.
- Every fermion must be paired with exactly one other, each of whose components are equal to one another in *all* basis vectors (though not necessarily equal to the corresponding components in any other basis vector). This condition essentially ensures that fermionizing a bosonic degree of freedom still has a sensible interpretation in terms of worldsheet bosons.
- Any fermion labeled with a subscript c forms a complex pair with the non-subscripted fermion of the same symbol (e.g. $\overline{\psi}_c^1$ is paired with $\overline{\psi}^1$, etc). This means that the compactified fermions on the left are free to form pairs with right-moving fermions, a case that will become important in analyzing the gauge group of the generated model.
- The x^i on the right are paired such that x^{2n+1} is paired with x^{2n+2} for all $n \in \mathbb{Z} \cap [0, 2]$
- The order of the left-moving fermions may be freely changed in the basis vectors, so long as the same pairings are maintained across all basis vectors (i.e. If $\overline{y}^1$ and $\overline{w}^1$ are exchanged in one basis vector, they must be exchanged in all basis vectors to ensure that the pairing of fermions is not disturbed). Such re-orderings will not affect the physical model produced.
- The right-moving sides of the basis vectors are of order 2.

In addition to these rules, the following conditions will ensure that a chosen set of basis vectors can produce a modular invariant model:

$$N_{ij}\vec{\alpha}_i^B \cdot \vec{\alpha}_j^B = 0 \pmod 8 \tag{35}$$

$$N_{ii}\vec{\alpha}_i^B \cdot \vec{\alpha}_i^B = 0 \pmod{16} \qquad \text{(For even-ordered basis vectors only)} \tag{36}$$

where N_{ij} refers to the least common multiple of N_i and N_j. Furthermore, all $\vec{\alpha}^B$ are expressed in a real basis and the dot product introduces a negative sign for the contribution from left-moving components. In addition we must have an even number of periodic components in any three basis vectors to ensure that it is possible to pair all real fermions properly.

Once we have built a valid set of basis vectors, we can construct the sectors of the model by taking all possible linear combinations of the form

$$\vec{\alpha} = \sum_{i=1}^{m} a_i \vec{\alpha}_i^B \qquad a_i \in \mathbb{Z} \cap [0, N_i - 1] \tag{37}$$

where the sum runs over the number of basis vectors. We will also convert the $\{\vec{\alpha}\}$ such that their values are in the range (-1, 1] for convenience in establishing later rules regarding these vectors. We can now give the explicit physical meaning of the components of the vectors $\{\vec{\alpha}\}$; namely the component $\vec{\alpha}^l$ gives the phase that the corresponding worldsheet fermion (f) picks up when transported around non-contractible loops in spacetime:

$$f \to e^{-i\pi\vec{\alpha}^l} f. \tag{38}$$

2. GSO coefficient matrix

We now have all the information we need to generate a valid set of basis vectors $\{\vec{\alpha}^B\}$, but we must also determine a valid matrix of coefficients (k_{ij}) which will be used to establish the GSO projection for the constructed model. There are three major rules for choosing these coefficients, which still grant considerable freedom in our choice of GSO projection. The first of these simply establishes that each column of the k_{ij}-matrix is of the same order as the corresponding basis vector:

$$N_j k_{ij} = 0 \pmod 2. \tag{39}$$

The other two constraints ensure that the GSO projection leads to a modular-invariant model:

$$\frac{1}{2}\vec{\alpha}_i^B \cdot \vec{\alpha}_j^B = k_{ij} + k_{ji} \pmod 2 \tag{40}$$

$$\frac{1}{4}\vec{\alpha}_i^B \cdot \vec{\alpha}_i^B - s_i = k_{ii} + k_{i1} \pmod 2 \tag{41}$$

where $s_i = 0$ if $\psi^1 = \psi^1_c = 0$ and $s_i = 1$ if $\psi^1 = \psi^1_c = 1$. In practice, this gives us a choice in $\frac{m^2-m}{2} + 1$ components of k_{ij} for a model with m basis vectors.

3. Generating states

After having constructed a set of sectors $\{\vec{\alpha}\}$ from the basis vectors as well as a valid GSO coefficient matrix, we must generate the set of possible fermionic charge vectors $\{\vec{Q}\}$. This is achieved by applying fermion raising or lowering operators to the worldsheet fermions in the vacuum states indicated by the sectors. Explicitly, potential charge vectors are given by

$$\vec{Q} = \frac{\vec{\alpha}}{2} + \vec{F} \tag{42}$$

where the components of $\vec{F}$ take on values -1, 0, or 1 depending on whether we are applying a lowering operator to a particular worldsheet fermion, leaving it unchanged, or applying a raising operator respectively [17] While this construction method allows us to construct the charge vectors for all massless physical states in a given model, this may not be terribly efficient if we are interested only in a few general features of the model. In particular, one reasonable goal for a string model would be to produce a supersymmetric spectrum with three chiral generations of matter and a phenomenologically realistic gauge group (e.g. $SU(5)$ or $S0(10) \times U(1)$, both of which have been suggested by Grand Unified Theory (GUT) models). In order to evaluate a string model based on these criteria, we would have no need to produce the entire spectrum. For example, the scalar particles would not be relevant to such analysis, and we never need to explicitly construct the graviton, which appears in all string models.

In light of this, we can apply a few general rules for what types of states can arise from a given sector if particular fermionic excitations are applied in order to build only those particles necessary to determine the model's phenomenological viability. These rules, developed by the Early Universe, Cosmology, and Strings (EUCOS) group at Baylor University, are presented at length in [15]. Here, we simply quote the results in order to use them in our later analysis of WCFFHS model gauge groups. We can predict the states coming from a particular sector via the following rules:

- A sector characterized by periodic right-moving spacetime boundary conditions will produce spacetime fermions
- Sectors of right-moving length-squared 0 will produce all spacetime bosons in the model
- Supersymmetric partners will arise from sectors with left-moving length-squared 0 and right-moving length-squared 8

Next, we turn to those rules which inform our choice of components for $\vec{F}$ in Equation 42. Importantly, the same operator must be applied to both fermions in a complex pair, but this is not necessarily the case with left-right pairings. Furthermore, as in the oscillator construction, we only wish to produce massless states, and here, the mass (on the right and left respectively) is given by [18]

$$m_R^2 = \frac{1}{4}\vec{Q}_R^2 - \frac{1}{2} \tag{43}$$

$$m_L^2 = \frac{1}{4}\vec{Q}_L^2 - 1 \tag{44}$$

Thus, this construction method need only produce charge vectors with length-squared 2 on the right and 4 on the left. Rules for what types of particles will be produced by various $\vec{F}$ (also developed by the EUCOS group) may now be employed to develop the most efficient construction process. In particular, we have

- Raising an internal fermion mode on a bosonic sector will produce a spacetime scalar
- Exciting a spacetime bosonic mode will produce states in the gravity supermultiplet or bosons resulting from the gravity supermultiplet in 10 dimensions due to compactification (See section III A)
- For left-moving fermions paired with right-moving fermions, applying lowering operators to each will produce physically identical states

Given these rules, the construction process can be streamlined to produce only those states we need to analyze a particular model. To this end, the construction process will avoid producing scalars, ignore the spacetime bosonic degrees of freedom [19], and only lower left-moving modes in left-right pairs.

With these rules, all relevant states may now be generated and included in the model or projected out via the GSO projection. In this construction, the GSO projection equation becomes

$$\frac{1}{2}\vec{\alpha}^B_j \cdot \vec{Q}_{\vec{\alpha}} = \sum_i k_{ji} a_i + s_j \pmod 2 \tag{45}$$

where $\vec{Q}_{\vec{\alpha}}$ is a charge vector built on a particular sector (α), a_i is as in Equation 37 for the given α, s_j is as in Equation 41 for the given $\vec{\alpha}^B_j$. The dot product is defined to have a negative contribution from the left-moving part, and any contributions from left-right paired fermions contribute with an additional factor of $\frac{1}{2}$. After a set of charge vectors has been selected by this projection, we have the low-energy spectrum and can begin analyzing the phenomenological features of a particular model.

While the charge lattice construction may seem more convoluted due to the number of rules used in the construction process, these rules mainly result from a desire for optimal efficiency– a necessity if we are to gain statistical information on any reasonable portion of the string landscape. Working with the charge vectors directly will also significantly simplify the analysis of the gauge content (particularly the $U(1)$ content) of these models.

III. ANALYZING WCFFHS MODELS

A. Identifying Gauge Groups

The first step in examining the phenomenological properties of a given model is to identify its gauge groups. Models without a realistic gauge group can be discarded, and the remaining models can be more closely scrutinized to determine whether or not they contain realistic spectra overall.

The process of identifying the gauge groups of a model begins by identifying the gauge bosons among constructed states. This is relatively straightforward since these states must have a spacetime vector index, and they should arise from bosonic sectors as defined in section II C 3. A state can gain a spacetime vector index by having right-moving spacetime fermionic charges equal to 1 ($\psi^1 = \psi^1_c = 1$) or through a spacetime bosonic excitation on the left. In the first case, we would have charge vectors of the form

$$\left(1, 1, \vec{0}^{18} || \vec{\beta}\right) \tag{46}$$

where $\vec{\beta}$ is some vector satisfying the masslessness condition $\vec{\beta}^2 = 4$. We will refer to this group of bosons as the left-moving gauge bosons. Notice that on the right-moving side, the spacetime fermionic coordinates take care of the masslessness condition for this group immediately, yielding 0's for the remaining coordinates. Similarly in the second case, the bosonic excitation would take care of the left-moving masslessness condition (as shown in Equation 22 for the oscillator construction), and these charge vectors would be of the form

$$\left(0, 0, \vec{\gamma} || \vec{0}^{44}\right) \tag{47}$$

where $\vec{\gamma}$ satisfies the masslessness condition $\vec{\gamma}^2 = 2$. Note that the spacetime fermionic coordinates must both take on value 0, since the bosonic excitation already provides a spacetime vector index. We will refer to this second group of bosons as the right-moving gauge bosons. While it turns out that the right-moving gauge bosons are not terribly relevant to this analysis, understanding why this is so requires a more thorough understanding of how the gauge groups are identified.

As is the case in gauge theory in general, the identified gauge bosons form an adjoint representation of the gauge groups. In order to further analyze the root structure, the appropriate scalar product on the root space (i.e. an appropriately defined inner product among these charge vectors) must be defined. An immediate possibility is the product defined for Equation 45. This, however, is not correct, as a result of the left-right pairing in these models. As described in [20], these left-right pairs (also known as necessarily real fermions) correspond to a truncation of the roots in the representation, since they project Cartan generators in the representation established by the gauge states out of the spectrum [21]. This is equivalent to the fact that the components of the charge vectors associated with the complex fermions are the eigenvalues of the gauge group representation. Thus, the left-right pairs do not contribute to the inner product.

To further narrow down the choice of inner product, we next examine the precise form of the left-moving and right-moving gauge groups and note that regardless of how any individual coordinate contributes to the inner product, these charge vectors are orthogonal, and thus the left-moving and right-moving groups cannot combine to form representations of any higher rank gauge group. In many cases the right-moving gauge states will have little or no impact on the LEEFT, but the left-moving gauge states would certainly give rise to a gauge group that would be observed experimentally [22]. Because of this, we turn our attention strictly to the left-moving gauge group, and the inner product is defined to run only over the left-moving degrees of freedom:

$$\vec{Q}_i \cdot \vec{Q}_j = \sum_{k:\text{Left}} \vec{Q}_i^k \vec{Q}_j^k \tag{48}$$

where k runs over the complex (non-left-right paired) fermions.

With the inner product defined, all necessary tools are now available to identify the gauge groups. Having restricted our attention to the left-moving group, we focus only on the left-moving charge vectors, and the positive roots (defined by convention to be those left-moving charge vectors whose first non-zero element is positive) are identified. This set is then split into mutually orthogonal sets; that is, sets whose elements have inner product 0 with the elements of every other set. This breaks the set up according to the separate groups whose tensor product gives the overall gauge group. Each set can then be further analyzed to identify the simple roots of the representation by removing those elements which can be written as a linear combination of other elements. Once the simple roots have been identified, it is a simple matter to construct the Cartan matrix for each set, which is defined to have elements

$$C_{ij} = 2\frac{\vec{Q}_i \cdot \vec{Q}_j}{\vec{Q}_i \cdot \vec{Q}_i}. \tag{49}$$

Since the Cartan matrix uniquely identifies a group, we can compare the constructed matrix to known Cartan matrices and determine the gauge group for a particular model. In reality, it is often unnecessary to produce the entire Cartan matrix, since the gauge group can be determined by other properties, such as the number of positive roots and the length of some of those roots. In fact, in the software used to obtain the results presented in section IV B, a more complex but also more efficient set of rules is employed to identify the gauge groups. These rules are presented in detail in [15].

With this analysis, significant progress has been made toward understanding the gauge content of a constructed model, but a significant hole remains to be considered. It has been established that the complex left-moving elements are eigenvalues of the group representation, and since the gauge states are in the adjoint representation, each gauge state corresponds to a generator of the group. The obvious question, then, is how to deal with the Cartan generators, for they will be associated only with zero-length roots. For the non-Abelian groups that can arise in these models, this does not pose an issue since the positive roots are sufficient to identify them, but the same cannot be said of the $U(1)$ groups. A given $U(1)$ group has only a single generator, which is also a Cartan generator (since the entire group is Abelian). As such, $U(1)$ groups, though they may appear in a model, will not have been detected by our present analysis, and we must turn to other means to identify them. The first goal will simply be to determine how many $U(1)$ groups are present in a model, and this will be achieved by determining the overall rank of the gauge group and comparing it to the rank of the identified non-Abelian groups.

B. Rank of the gauge group

In order to determine the rank of the gauge group for a particular model, we begin by determining the maximum possible rank and then examining how that rank is reduced. For a 10-dimensional model, the maximum rank of the gauge group will be 16, corresponding to the 16 dimensions of the even self-dual lattice for the bosonic string. This is in accordance with the rank of the two consistent overall gauge groups for 10-dimensional heterotic theories: $SO(32)$ and $E_8 \times E_8$. Next, the effect of compactification on the rank must be considered.

From standard Kaluza-Klein reduction of dimensions, it can be shown that each additional compactification of a large spacetime dimension contributes an additional $U(1)$ group from the 10-dimensional metric as well as a $U(1)$ from the antisymmetric tensor, each of which could potentially be promoted to some other groups in combination with the other gauge states in the model [23]. The $U(1)$'s arising from the antisymmetric tensor, however, correspond to the right-moving gauge states discussed in section III A, so the maximum rank of the left-moving gauge group with d compactified dimensions is$16 + d$.

Finally, we consider potential sources of rank reduction or "rank-cutting" in a WCFFHS model. Since the rank of a group is exactly equal to the number of Cartan generators in that group, rank-cutting will occur when the gauge state or states associated with one or more of those generators is projected out of the spectrum. As discussed in section III A, this will occur in the case of necessarily real fermions, and, indeed, there is no other mechanism in this construction to achieve this effect[24]. Thus, the overall rank of the gauge group is

$$16 + d - N_{\mathrm{LR}} \tag{50}$$

where N_{LR} is the number of left-right pairs in the given model.

Having established the rank of the overall gauge group, the number of $U(1)$'s can now be determined. Since each $U(1)$ increases the rank by 1, the number of $U(1)$ groups $\left(N_{U(1)}\right)$, is given by

$$N_{U(1)} = 16 + d - N_{\mathrm{LR}} - R_{\mathrm{NA}} \tag{51}$$

where R_{NA} is the rank of the non-Abelian gauge groups. While this does not yield a complete picture of the gauge interactions, the full gauge group has at least been determined, and we can proceed to examine the remaining properties of the spectrum.

C. FF_Framework

While the calculations involved in the charge lattice construction (and the analysis of its spectrum) described thus far are not excessively challenging, the sheer number of possible states within a given model as well as the overwhelming number of possible models in the string landscape necessitates an automated method for producing and analyzing these models. In particular, gaining any meaningful statistical information on a portion of the landscape will require extremely fast implementations of efficient construction and analysis algorithms. To this end, the EUCOS group at Baylor University has developed a fast, highly extensible C++ framework known as "FF_Framework" (Free Fermionic Framework) which implements the charge lattice method for constructing and analyzing WCFFHS models, and the results presented in section IV B will make considerable use of this tool[25].

A number of initial studies have been carried out using this framework, and many of the results have been summarized in [15]. Prior to the research presented herein, however, FF_Framework did not include a method for deriving anything more about the Abelian content of the constructed models than the number of $U(1)$ groups. Given that these groups can have a significant impact on the phenomenology of the models, we now address the issue of the $U(1)$ gauge content directly.

IV. $U(1)$ GAUGE GROUPS IN WCFFHS MODELS

In the Standard Model, the $U(1)$ gauge symmetry, in combination with the $SU(2)$ group underlies our understanding of electroweak interactions, and it is associated with the conserved charge known as weak hypercharge. Thus, an understanding of the $U(1)$ groups in a string model is necessary to arrive at a complete picture of the gauge interactions in that model. In particular, we must derive how each of the matter states in a given model are charged under the $U(1)$ gauge states in order to understand their interactions.

For a non-Abelian group, this process is straightforward; we simply take the inner product of the left-moving parts of the matter state and the gauge states (as defined in Equation 48). As established in section III A, however, the outlined construction process does not generate the $U(1)$ gauge states, so another algorithm must be employed to find the $U(1)$ charges. In particular, although the $U(1)$ gauge states cannot be directly constructed, it is possible to construct a $U(1)$ gauge generator with no net charge (as expected for the $U(1)$ gauge state), but the sum of whose charges squared satisfies the masslessness condition. That is to say, we are looking for a state of the form

$$\left(1, 1, \vec{0} || \vec{\beta}\right) \tag{52}$$

where $\vec{\beta}^2 = 4$ to satisfy the masslessness condition and the entire charge vector has zero net charge. The dot product of the matter states with these gauge generators will yield the desired $U(1)$ charges (again using the product defined in Equation 48).

A. Constructing $U(1)$ gauge generators[26]

In order to determine the $U(1)$ generators, we first consider what constraints we have on the form of these vectors. In particular, being gauge generators, the right-moving parts are determined just like those of the other gauge states in the model; in particular, their spacetime-fermionic components will both take on value 1 while the rest of the components take on value 0. Furthermore, in order to be the gauge generators for a separate $U(1)$ group, the left-moving part of these states must be orthogonal to the simple roots of the other gauge states in the model as well as to the gauge generators of other $U(1)$ groups. This is the *only* condition that these generator states need satisfy however, for they are unique only up to rotations in the space orthogonal to the simple roots of the non-Abelian groups[27]. That is to say, the constraints on the form of a $U(1)$ gauge generator that ensures an anomaly-free model (ignoring the possibility of a single anomalous $U(1)$ group to be addressed in section V) is also satisfied by any linear combination of the generators, so any set of generators that satisfy the orthogonality condition will provide a consistent theory (given, of course, that the construction method itself produces a consistent model, which it always does)[28].

Given this fact, the problem of constructing the $U(1)$ gauge generators reduces to a straightforward linear algebra problem. Specifically, we must find a set of mutually orthogonal vectors (the left-moving part of the $U(1)$ gauge generators) which are also orthogonal to a given set of linearly independent vectors (the simple roots of the non-Abelian gauge groups)[29]. This mutually orthogonal set of vectors will be denoted $\mathbf{V}_{U(1)}$ and the given set of linearly independent vectors will be denoted $\mathbf{V}_{\mathrm{SR}}$. The cardinality of $\mathbf{V}_{U(1)}$ is given in Equation 51, and the cardinality of $\mathbf{V}_{\mathrm{SR}}$ will be denoted N_{SR}. For convenience and computational efficiency, all complex fermions will be denoted by a single element in these vectors.

The first priority for this algorithm is to generate so-called external gauge states, which have all left-moving elements 0 except for a single element, to which we can assign the value 1. These states will be constructed whenever there is a particular coordinate which takes on value 0 for every vector in $\mathbf{V}_{\mathrm{SR}}$. The number of elements of $\mathbf{V}_{U(1)}$ constructed up to any given point in the algorithm will be denoted $n_{U(1)}$.

Once the external gauge states have been constructed, the remaining $N_{U(1)} - n_{U(1)}$ states orthogonal to $\mathbf{V}_{\mathrm{SR}}$ and the $U(1)$ gauge generators must be constructed. Consider the next vector $\vec{v}$ to be constructed in $\mathbf{V}_{U(1)}$. If the union of $\mathbf{V}_{\mathrm{SR}}$ and the elements of $\mathbf{V}_{U(1)}$ constructed thus far is denoted $\mathbf{U}$, $N_{\mathrm{SR}} + n_{U(1)}$ constraint equations can be constructed for the elements of $\vec{v}$ by setting the dot product of $\vec{v}$ with every element of $\mathbf{U}$ equal to 0. This provides a system of linear equations, whose solution yields the components of $\vec{v}$, but notice that this system is underdetermined. Specifically, there will be l free variables, where l is given by

$$l = N_{U(1)} - n_{U(1)}. \tag{53}$$

Thus, arbitrary values may be assigned to l components of $\vec{v}$ [30]. Importantly, however, while the assigned values are arbitrary, the choice of which components receive those values is not. In particular, if the vectors of $\mathbf{U}$ are not linearly independent in the subspace determined by the remaining, unassigned components, the resulting system will be overdetermined and likely inconsistent.

To illustrate this point, consider the simple problem of finding a vector $\vec{w}$ orthogonal to both of the following:

$$\vec{u}_1 = (2, 1, 1) \tag{54}$$

$$\vec{u}_2 = (1, 2, 2)\,. \tag{55}$$

The components of $\vec{w}$ are denoted by

$$\vec{w} = (w_1, w_2, w_3)\,. \tag{56}$$

In the system of equations determining these elements, there will be one free variable, so one component of $\vec{w}$ can receive an arbitrary value. Notice, however that if we choose to assign an arbitrary value (say, 1) to w_1, $\vec{u}_1$ and $\vec{u}_2$ are not linearly independent in the subspace determined by their second and third components. Thus, this choice will result in the following (clearly inconsistent) system of equations:

$$w_2 + w_3 = -2 \tag{57}$$

$$w_2 + w_3 = -\frac{1}{2}. \tag{58}$$

Therefore, in solving for the elements of a vector $\vec{v}$ in $\mathbf{V}_{U(1)}$, care must be taken in the choice of components to which arbitrary values will be assigned.

In order to ensure that this choice of components yields a consistent system of linear equations, we construct a matrix $\mathbf{M}$, whose rows correspond to the vectors of $\mathbf{U}$. Then, the Gaussian elimination algorithm is applied to this matrix, and any column not containing a pivot is identified (where a pivot is defined to be the first non-zero element of a row if that element's column does not contain the first non-zero element of any prior row). Arbitrary values (for convenience chosen to be 1) can then safely be assigned to the corresponding elements of $\vec{v}$, since $\mathbf{U}$ will be linearly independent in the remaining components, and the remaining elements of $\vec{v}$ can be determined by solving the resulting system of linear equations. In the C++ implementation developed to extend FF_Framework, Gauss-Jordan elimination is used for this purpose in order to improve readability and maximize code reuse, but any common solution method can be applied. After the components of $\vec{v}$ have been determined, $\vec{v}$ is added to $\mathbf{U}$, and the process is repeated until all $N_{U(1)}$ vectors of $\mathbf{V}_{U(1)}$ have been found, thereby completely determining the $U(1)$ gauge generators. The charges of the matter states under these $U(1)$'s may then be determined via the inner product of the matter states with the gauge generators.

B. Initial results [31]

Having developed an algorithm for finding the $U(1)$ charges in a WCFFHS model, we next examine what new information can be obtained about the WCFFHS landscape. In order to do this, the algorithm was implemented as a C++ module in FF_Framework, emphasizing speed and extensibility. This implementation was then used in conjunction with the rest of FF_Framework to construct 1,425,976 models and analyze their properties. In particular, we are interested in analyzing the effect of the $U(1)$ charges on models with potentially interesting or realistic phenomenology. To this end, the models constructed were four-dimensional, and each had three basis vectors: the universal $\mathbb{1}$ vector, a vector known as the supersymmetric (SUSY) vector of the form $\left(1, 1, (1, 0, 0)^6 \,||\, (0)^{44}\right)$, and a vector of the form $\left((0)^{20} \,||\vec{\lambda}\right)$. Models built using only a basis vector of this form in addition to the $\mathbb{1}$ and SUSY vectors are known as gauge models. The basis vectors were also chosen to be of order 22 or less. In addition, this run was confined to models which previous analysis had shown would contain at least a single copy of the Standard Model gauge group. While not intended to be an exhaustive search of even a subset of the WCFFHS landscape, this run provides some initial insight into how the $U(1)$ charges affect our overall understanding of a certain class of potentially phenomenologically interesting models, and some of these results are presented herein.

1. Gauge content

While all of the produced models are potentially of phenomenological interest (given that they contain the Standard Model group), models containing groups in proposed GUT models may be of even greater interest. In particular, we consider the minimal left-right model, the Georgi-Glashow model, the flipped $SU(5)$ model, the Pati-Salam model, the trinification model, and the $SU(6)$ model. In all, 67.6% of constructed models contained groups proposed by one of these GUT models. The proportion of constructed models containing the gauge

GUT Model(s)	Gauge Group	Proportion
Minimal Left-Right Model	$SU(3) \times SU(2) \times SU(2) \times U(1)$	0.61
Georgi-Glashow Model/ Flipped $SU(5)$	$SU(5)$	0.32
Pati-Salam model	$SU(4) \times SU(2) \times SU(2)$	0.17
Trinification model	$SU(3) \times SU(3) \times SU(3)$	0.066
$SU(6)$ model	$SU(6)$	0.024

TABLE I. Proportion of all models constructed containing proposed GUT symmetry groups. Note that Georgi-Glashow and the flipped $SU(5)$ model have been grouped together since every model constructed contains at least one $U(1)$ and the gauge groups of these two GUT models differ only by a $U(1)$.

group proposed by each of these models is displayed in Table I. Note that these models will contain additional groups beyond these GUT groups, and some contain multiple proposed GUT groups (leading to a total proportion over 0.676).

From this table, we see a significant prevalence of models with phenomenologically interesting gauge content, so we turn now to the $U(1)$ groups specifically. Models in this run had between 3 and 8 $U(1)$'s with a median value of 4. Given that prior studies had not generated $U(1)$ content, one question of particular interest for this run was how much the $U(1)$ content would affect prior results obtained via this construction method. To this end, the number of $U(1)$'s in each model was compared to the number of each other group in the models to see if there was any correlation between the presence of a particular group and the presence of $U(1)$'s. The highest correlation obtained was between the number of $U(1)$'s and the number of $SU(4)$'s as shown in Figure 1, but even this correlation was extremely weak, with an R^2 value of just 0.49. Thus, there seems to be no immediate statistical rule that can be used to predict how many $U(1)$'s (and thus, in part, how significant a correction may be required for a particular model whose properties have been analyzed without considering $U(1)$ content) will be present based on the presence or absence of other gauge groups.

2. Matter state uniqueness

Next we turn to issues related to the uniqueness of models. In general, determining whether or not two constructed WCFFHS models are unique is a non-trivial problem [32],

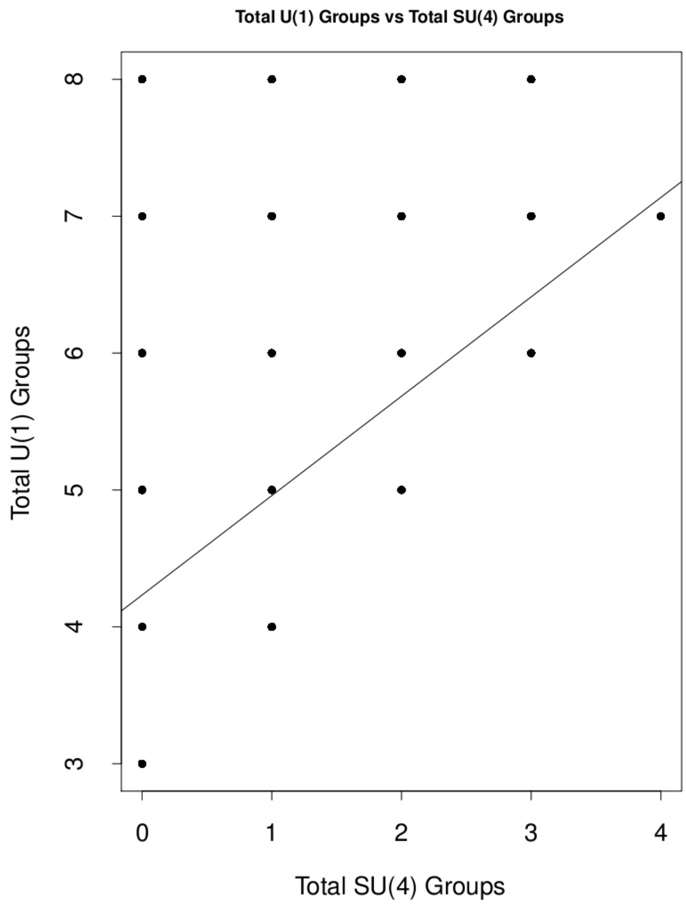

FIG. 1. Total $U(1)$ groups in constructed models vs. their total number of $SU(4)$ groups, the most significantly correlated relationship between the number of $U(1)$'s and the number of any other group

but it is an important issue to address in relation to the challenge of the string landscape. In particular, if it can be shown that significant swathes of the landscape correspond to a single physical model, it is possible that the $\sim 10^{500}$ different models of the landscape could be cut down to a more manageable number.

Construction of $U(1)$ content can be used to revisit prior results on model uniqueness in a few ways. One possibility is that two models previously thought to be identical could be shown to be different through a comparison of the exact relationship among the $U(1)$ generators and other gauge states in the model. In general, however, this comparison will be extremely difficult, since the $U(1)$ generators are unique only up to rotations in a subspace as shown in section IV A, and this problem will therefore be left for future research.

A slightly easier issue to address is that of matter state uniqueness (where uniqueness is defined in relation to the gauge charges) *within* a given model [33]. In particular, matter states that were identical in their non-Abelian charges may differ in their $U(1)$ charges. Although consideration of the $U(1)$ charges still does not allow comparison of matter content between models, counting the number of unique matter states may occasionally allow a distinction to be made among models previously thought to be identical simply by recognizing that the models have unequal numbers of unique matter states.

In light of this, we now consider what insights the present data can provide on the uniqueness of matter states within a single model. In particular, we consider the quantity B, defined to be the ratio between the number of matter states in a given model unique in their non-Abelian charges to the number of matter states unique in their $U(1)$ charges, for this ratio will give a solid measure of the error that prior studies would have produced in counting unique matter states within any given model. The distribution of B for the run of approximately 1.4 million models is displayed in Figure 2. As we can see from this figure, B was equal to 1 for most models, indicating that the inclusion of $U(1)$ charges had no effect on the number of unique matter states. In a significant minority of models, however, B fell below 1, and it reached a minimum of just 0.2083, indicating an error of approximately 80% in the counting of unique matter states without considering the $U(1)$ content.

Given this distribution, hoping for a correlation between the presence of certain non-Abelian groups and the value of B is not unreasonable, since this would allow us to predict (on average) whether a given model would fall into the majority of cases where $U(1)$ gauge content does not affect matter state uniqueness or into the troubling minority where it does. Unfortunately, however, analysis of the run under consideration showed no significant correlation between the number of any non-Abelian gauge group in a model and the value of B. Indeed, the most significant correlation was between B and $SU(10)$, which had an R^2 value of just 0.07.

While B gives us some initial insight into the effect of $U(1)$ charges on matter state uniqueness, other statistics can also shed some light on what makes these states unique. In particular, consider the quantity G, defined to be the ratio between matter states unique in their $U(1)$ charges alone to matter states unique in their non-Abelian charges alone, which will give further information on how $U(1)$ charges determine matter state uniqueness. The

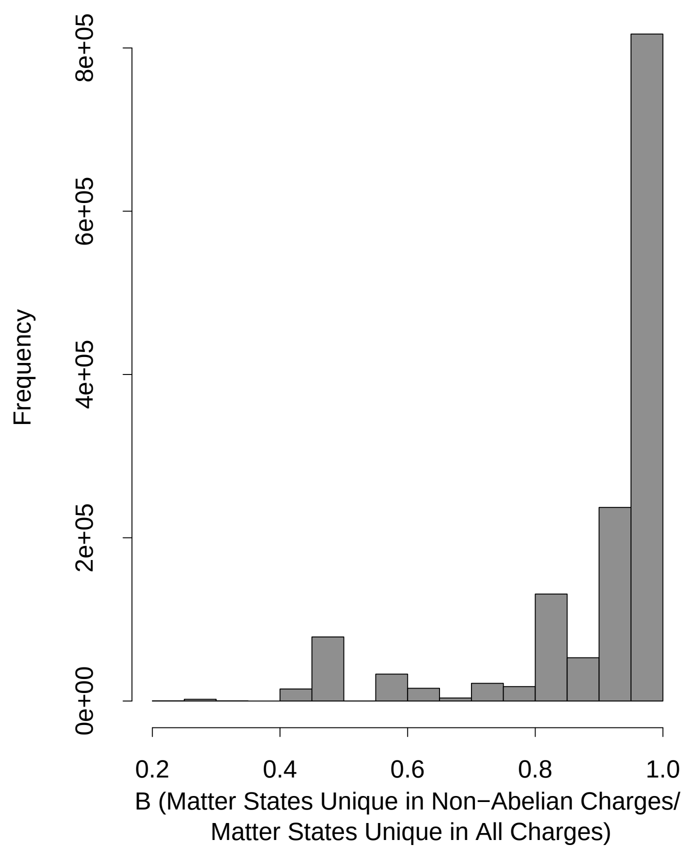

FIG. 2. Distribution of B, which gives a measure of the error in counting unique matter states as a result of not considering their $U(1)$ charges

distribution for G is given in Figure 3. While this distribution is sharply peaked below 1 (indicating that a significant number of models contained matter states that were identical in $U(1)$ charges but unique in non-Abelian content), note that a significant minority of models had $G > 1$, indicating that they contained matter states whose uniqueness was determined solely by their $U(1)$ charges. This adds an interesting complexity to the relationship between gauge charges and matter state uniqueness, and further research could investigate the origin of each of these classes of matter states.

Having considered the statistical information relating the $U(1)$ content of WCFFHS models and matter state uniqueness, we finally consider the larger results for matter state uniqueness given the benefits of the new algorithm for determining $U(1)$ charges. The overall dis-

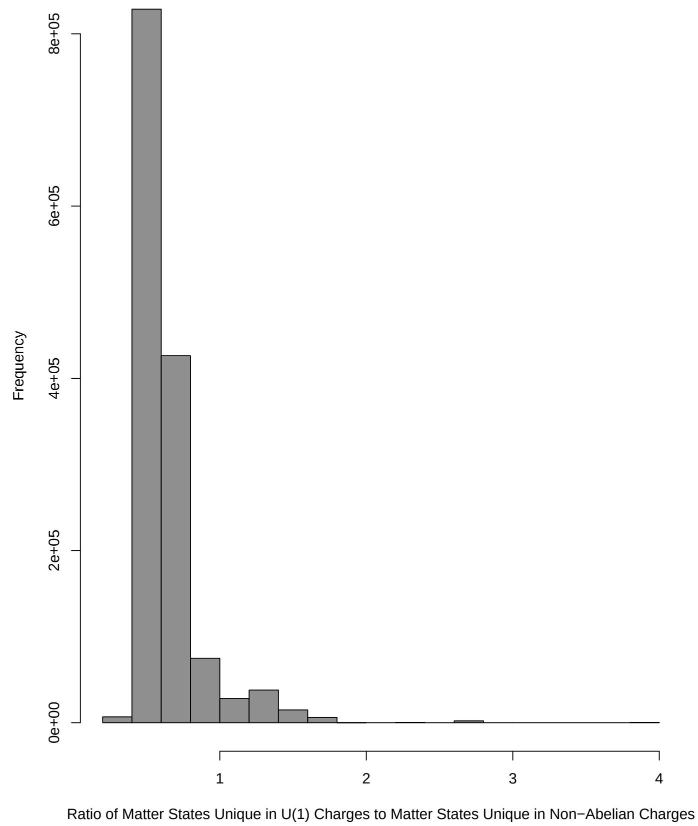

FIG. 3. Distribution of G, which shows the degree to which the $U(1)$ charges as opposed to the non-Abelian charges are responsible for determining matter state uniqueness

tribution for the number of unique matter states is given in Figure 4. This distribution is fairly symmetric with a distinct peak at 12 unique matter states, the median for these data. A Shapiro-Wilks normality test shows that these data do not follow a normal distribution, however.

While the data from this run of WCFFHS models begins to provide some insight into the effect of $U(1)$ groups on this type of model, the presence of an anomalous $U(1)$ group in a constructed model can have even more profound effects than those discussed thus far. We now turn our attention to the mechanisms behind these changes.

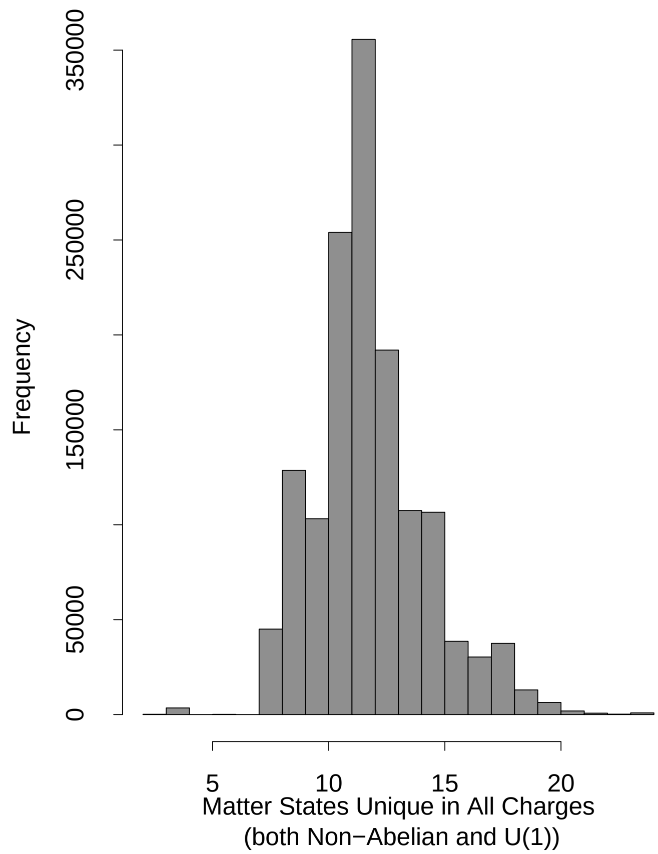

FIG. 4. Distribution of the number of unique matter states, taking into account the new information provided by the $U(1)$ charges

V. ANOMALOUS $U(1)$ GROUPS

In addition to completing the picture of gauge interactions for a particular model, analysis of the $U(1)$ charges can have a significant impact on the LEEFT in other ways. In particular, the presence of an anomalous $U(1)$ group in a model can both affect the unbroken gauge groups in the LEEFT and change the low-energy spectrum, giving vacuum expectation values to some fields at high energies and thereby removing them from the low-energy theory. In order to understand this mechanism, we first consider the larger issue of anomalous groups in string theory.

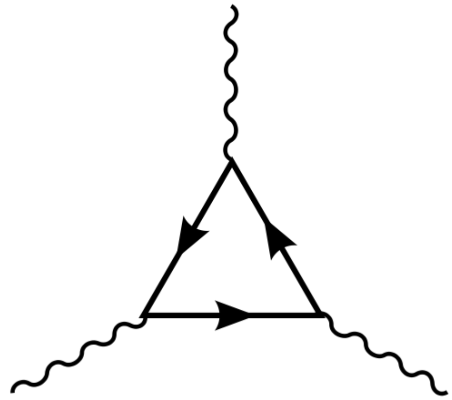

FIG. 5. A Feynman diagram of the type giving rise to gauge anomalies in 4 dimensions. The straight lines here refer to chiral fermions, which are shown coupling to gauge bosons

A. The Green-Schwarz mechanism

In gauge theories generally, we encounter the possibility that a given gauge symmetry is broken by quantum corrections beyond tree-level such that it cannot be restored by adding additional terms to the low-energy effective action [34]. The presence of such an anomaly is extremely discouraging in any theory that purports to describe a physical system, since these gauge symmetries are required in order to ensure a consistent quantum theory and avoid states with negative norms (as mentioned in section I A). Generally, a gauge theory containing such anomalies must be rejected outright, and indeed, the apparent inability of string theory to produce a non-anomalous theory with chiral fermions led many researchers to abandon it entirely in the early 1980's. In 1984, however, a method known as the Green-Schwarz mechanism was discovered, which permitted cancellation of the anomalies that had appeared so problematic. While a complete derivation of the Green-Schwarz mechanism may be found in any introductory string textbook, the mechanism is touched on here to provide background for a more interesting specific case of anomaly cancellation.

In four dimensions, we are concerned with the possibility of gauge anomalies arising due to corrections from diagrams of the type indicated in Figure 5. In ten dimensions, this central triangle is replaced by a hexagon coupled to gauge bosons at each vertex. We consider a gauge field $\mathbf{A}$ with field strength given by the 2-form [34]

$$\mathbf{F} = \frac{1}{2}\sum_{\mu\nu} F_{\mu\nu} dx^{\mu} \wedge dx_{\nu} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}. \tag{59}$$

In addition, we have the Ricci tensor $\mathbf{R}$. In 10 dimensions, the anomaly appears as a linear combination of terms composed of the field strength and Ricci tensor. In the Green-Schwarz mechanism, however, we recognize that an additional term can be added to the action whose variation precisely cancels the anomaly, a cancellation that is only possible because of the gauge-invariant combination of the field strength and the Ricci tensor that appears in the 10-dimensional anomaly for particular overall gauge groups. While this cancellation is delicate, requiring both the correct dimensionality and the correct gauge groups, it provides a mechanism to eliminate these anomalous groups from a string model and yield a viable quantum field theory.

In addition to the anomalies handled by the Green-Schwarz mechanism, however, another type of anomalous $U(1)$[35] can appear in WCFFHS models. In order to deal with these, we must turn to a different (though related) mechanism for handling this type of anomaly [36].

B. The Dine-Seiberg-Witten mechanism

In order to understand the mechanism for dealing with the anomalous $U(1)$'s coming out of the charge lattice construction, we consider a particular term of the overall Lagrangian for a string model. This term is known as the Fayet-Iliopoulos D-term, which gives the contribution of an auxiliary field D to the Lagrangian in terms of the dilaton field ϕ, scalar fields c_i, and the charges of those scalar fields under a $U(1)$ in the model. Such a term will only appear if the trace of the $U(1)$ charges over the matter states is non-zero, and it is given by [37]

$$\phi^{-2}D^2 = \phi^{-2}\left(\phi^2 + \sum_i e_i c_i^* c_i\right)^2 \tag{60}$$

which gives us explicit mass terms for the c_i. In general, however, since e_i can take on either sign, this would make some of these fields tachyonic. To avoid this unphysical situation, we instead consider the possibility of providing a vacuum expectation value to some of the c_i. This is most straightforwardly illustrated by considering the case of a single c with negative charge e under the $U(1)$ in question. In this case, the potential contribution at this order for ϕ can be written as

$$V = \phi^4 + 2ec^*c\phi^2 + e^2(c^*c)^2. \tag{61}$$

Setting the derivative with respect to c^* equal to 0 (in order to solve for the vacuum expectation value of c) and rearranging, we have

$$e\langle c\rangle^2 = -\phi^2. \tag{62}$$

Since e is negative, this gives a real vacuum expectation value for c, thereby removing it from the LEEFT. This shift in the vacuum takes care of the anomaly that would otherwise be introduced for such a $U(1)$. In general, a more complex pattern of vacuum expectation values can be given to all of the scalar fields in a theory in order to keep the superpotential zero (as well as its derivative) while also eliminating the anomaly. Notice also that this mechanism can only be applied to a single $U(1)$, for, as pointed out in [37], if a second such $U(1)$ existed, its D-term would take on the form

$$\phi^{-2}D'^2 = \phi^{-2}\left(1+\sum_i e'_i c_i^* c_i\right)^2, \tag{63}$$

which would give the scalar fields a tree-level mass. Nevertheless, we have seen that the presence of a single anomalous $U(1)$ can significantly affect the LEEFT for a constructed string model, so we finally turn to the issue of dealing with such anomalies in the charge lattice construction.

C. Anomalous $U(1)$'s in the charge lattice construction

Although the problem of deriving an efficient algorithm for adjusting the constructed spectrum in the presence of an anomalous $U(1)$ has been left to future research, we consider initial steps that will be necessary to do so. In particular, as described in section V B, the Dine-Seiberg-Witten mechanism may only be applied to a single anomalous $U(1)$ group, so we must develop tools to identify anomalous $U(1)$'s and deal with cases where more than one $U(1)$ appears to be anomalous. Both of these tasks are very straightforward.

Since anomalous $U(1)$'s will have a non-zero trace over the matter states (which we will denote by T_i for each $U(1)_i$ in the model), we simply take the sum of charges of the matter states under a particular $U(1)_i$ and check to see if that sum is non-zero. If it is, this $U(1)_i$ is anomalous. Given that the $U(1)$ gauge generators are unique only up to rotations in a subspace, however, there is no guarantee that only a single $U(1)$ will be anomalous. In fact, if a model contains an anomalous $U(1)$, most of its constructed $U(1)s$ will appear anomalous. At this point, however, we once again take advantage of the non-uniqueness of these gauge generators and rotate them such that only one remains with a non-zero trace. The left-moving part of the one “true” anomalous gauge generator $\vec{v'}_X$ is given by

$$\vec{v'}_X = \sum_i T_i \vec{v}_i \tag{64}$$

where the v_i are the left-moving components of the old set of gauge generators. The remaining generators can then be obtained by adding $\vec{v'}_X$ to the set $\mathbf{V}_{\rm SR}$ in the notation of section IV A and then proceeding with the $U(1)$ gauge generator construction method as usual.

Using these techniques, we obtain a single anomalous $U(1)$ group, to which the Dine-Seiberg-Witten mechanism may be applied, thereby eliminating the anomaly altogether and producing a shift in the allowed vacuum states. As we can see from the run of 1.4 million WCFFHS models described in section IV B, this is quite important, for in this run, only 57 models contained no anomalous $U(1)$ groups, and of these, only 5 contained a GUT group. This should not dismay us at all, however, for the presence of an anomalous $U(1)$ can be phenomenologically favorable, particularly if it results in a vacuum expectation value for exotic states in the model.

VI. CONCLUSIONS

The potential for analyzing the string landscape using WCFFHS construction techniques is significant. Using these techniques, we can examine the properties of potential string models in considerable detail and systematically search for models of phenomenological interest. In order to get a complete sense of the gauge interactions for a model and understand the models low-energy spectrum, however, we must continue to develop efficient tools for constructing and analyzing all gauge content, including the $U(1)$ gauge content, for these models. To this end, the tools described herein will hopefully prove useful in future systematic searches of the landscape. Further research could improve these tools in a number of ways, however.

In particular, an efficient and rigorous method of determining equivalence of models is essential in order to determine whether or not the search space on the landscape can be reduced at all. Furthermore, while this equivalence will be fairly difficult to establish, it is possible that establishing matter state uniqueness between models could be made easier by developing an effective method for comparing sets of gauge generators among constructed models. Finally, while a technique for rotating the $U(1)$ gauge generators to leave only a single anomalous $U(1)$ was presented in section V C, future developments could attempt to build the $U(1)$ gauge generators this way without requiring a subsequent recalculation of those generators. This would increase efficiency, leaving more computing time for analysis of the resultant shifted vacuum. While much work remains to complete the picture of WCFFHS models constructed using the methods described, the tools already available offer hope that WCFFHS construction may continue to further our understanding of the string landscape and perhaps point to the elusive string model that matches our own experiences.

ACKNOWLEDGMENTS

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[3] R. J. Szabo, *An Introduction to String Theory and D-brane Dynamics* (Imperial College Press, London, 2004).

[4] Here, strictly bosonic string theory is ignored because it has some features that make it phenomenologically unattractive. In particular, perturbative solutions in bosonic string theory can generally admit tachyonic states. This can lead to an instability through tachyon condensation.

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[9] J. Bagger, D. Nemeschansky, N. Seiberg, and S. Yankielowicz, Nucl.Phys. **B289**, 53 (1987).

[10] H. Kawai, D. C. Lewellen, and S. H. Tye, Nucl.Phys. **B288**, 1 (1987).

[11] I. Antoniadis, C. Bachas, and C. Kounnas, Nucl.Phys. **B289**, 87 (1987).

[12] This section follows the original formulation in [10] fairly directly. Please see this paper for more extensive derivations of the constraints on the partition function.

[13] F. Gliozzi, J. Scherk, and D. I. Olive, Nucl.Phys. **B122**, 253 (1977).

[14] Note that we would generally have to take $\overline{n_l^q}$ into account as well if we were working in a complex basis.

[15] T. Renner, *Initial Systematic Investigations of the Weakly Coupled Free Fermionic Heterotic String Landscape Statistics*, PhD dissertation, Baylor University, Department of Physics, Baylor University, Waco, TX, 76798-7316, USA (July 2011), `https://beardocs.baylor.edu/xmlui/handle/2104/8240`.

[16] This notation precisely follows that presented in [15].

[17] Note that these operators are with respect to the fermionic *charges*. We would have to work backward from the derived charge vector to determine how these operators affect the underlying oscillator state. In general, applying a lowering operator to the sectors is a valid operation in this formulation.

[18] In these formulae, the spacetime bosonic excitations are temporarily ignored for reasons that will be stated explicitly during analysis of the gauge content of the models in section III A.

[19] While the gravity supermultiplet is not terribly relevant to this type of analysis for fairly straightforward reasons, it should be less obvious why the bosons resulting from compactification do not come into play. This problem will be explored in further detail in section III A.

[20] K. R. Dienes and J. March-Russell, Nucl.Phys. **B479**, 113 (1996), arXiv:hep-th/9604112 [hep-th].

[21] This is best understood by recognizing that the Cartan generators occur due to states with excitations of both modes of a complex fermion pair, as discussed in [20]. Establishing a left-right pairing eliminates a possible complex pair in the model, thereby projecting out a Cartan generator.

[22] Special thanks to Douglas Moore of Baylor University for pointing out that the right-moving gauge states would be removed from the spectrum for $N = 0, 1$ supersymmetry. At $N \geq 2$, it is a general feature of supersymmetry that the spectrum does not contain chiral matter and is therefore not of phenomenological interest.

[23] J. Polchinski, *String Theory: Superstring Theory and Beyond*, Vol. II (Cambridge University Press, Cambridge, 1998).

[24] This is not to say that the rank cannot be reduced in the LEEFT through symmetry-breaking, however. In particular, if the constructed model contains an anomalous $U(1)$, some of the gauge groups can be broken at high energy scales, giving an effective rank-cut for the LEEFT. See [28], for instance.

[25] Please see `http://homepages.baylor.edu/eucos/` for more details on this framework.

[26] The algorithm described herein was developed by the author in collaboration with the EUCOS group and Lesley Vestal (formerly of the University of British Colombia) as part of a Research Experience for Undergraduate program funded by NSF grant PHY-1002637. This algorithm and some initial results found by applying it were first presented in [38].

[27] Note that throughout this section we are ignoring the issue of normalization for these generator states, as it will not be relevant to our present analysis and is therefore computationally inefficient.

[28] T. Kobayashi and H. Nakano, Nucl.Phys. **B496**, 103 (1997), arXiv:hep-th/9612066 [hep-th].

[29] Note that orthogonality here is defined according to the inner product in Equation 48.

[30] Notice that this implies $\mathbf{V}_{U(1)}$ is unique only up to rotations in an $N_{U(1)}$-dimensional space as expected.

[31] A few of the results on matter state uniqueness and correlation among the presence of various gauge groups were originally presented in [38]. They are presented again here for completeness.

[32] See [15] for an in-depth discussion of the challenges involved with comparing WCFFHS models for uniqueness.

[33] Here, the term "matter state.

[34] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, 2007).

[35] Note that although we refer to these groups as anomalous throughout the current paper, the underlying theory, as we shall see, is not actually anomalous. Because of this, some phenome-

nologists prefer to refer to these groups as "pseudo-anomalous".

[36] K. R. Dienes and C. Kolda(2010), arXiv:hep-ph/9712322v1.

[37] M. Dine, N. Seiberg, and E. Witten, Nucl.Phys. **B289**, 589 (1987).

[38] W. Hicks, L. Vestal, J. Greenwald, D. Moore, T. Renner, *et al.*(2011), arXiv:1108.4082 [physics.comp-ph].